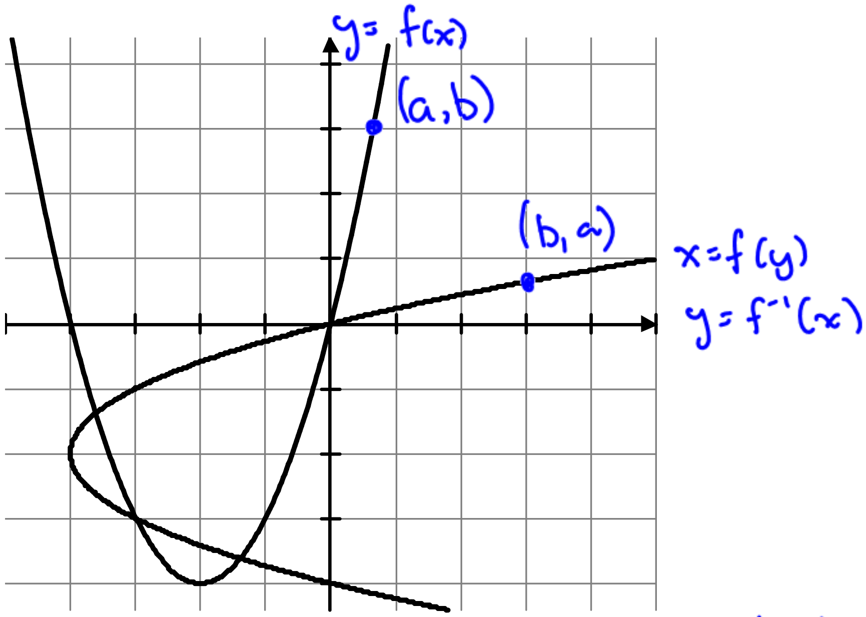
Derivative of the Inverse of a Function

What is the relationship between the derivative of a function and the derivative of its inverse?



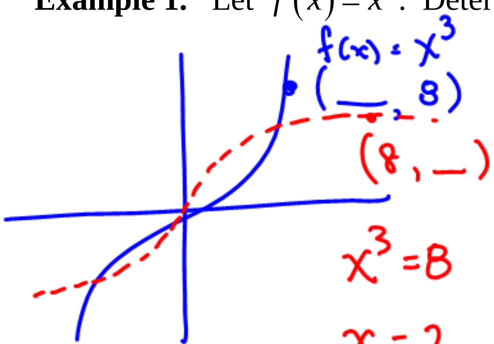
$g(x) = f^{-1}(x)$

$f(x)$	$g(x)$
(a,b)	(b,a)

the slopes of the tangents at these points are reciprocals.

$$g'(b) = \frac{1}{f'(a)}$$

Example 1. Let $f(x) = x^3$. Determine $\left.\frac{df^{-1}}{dx}\right|_{x=8}$



$x^3 = 8$

$x = 2$

$$= \frac{1}{f'(2)}$$

$$= \left.\frac{1}{3x^2}\right|_{x=2}$$

$$= \frac{1}{12}$$

Example 2. Let $f(x) = x^3 + x - 2$ and let $g(x)$ be the inverse function. Evaluate $g'(0)$

$g(x)$ has $(0, \_)$

$f(x)$ has $(\_, 0)$

solve for x $\quad 0 = x^3 + x - 2$

graphically

$x = 1$

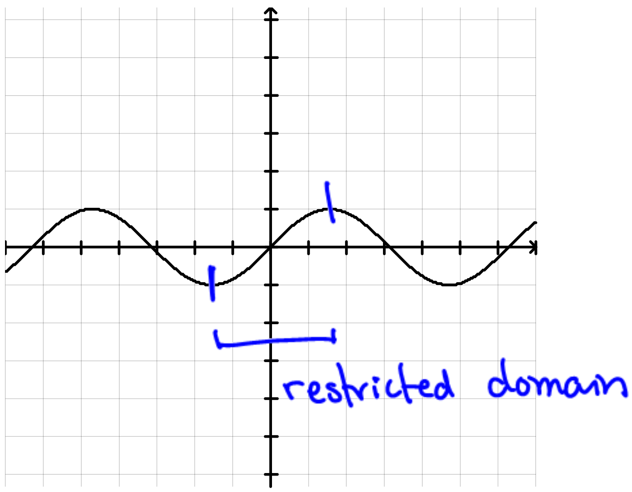
$$g'(0) = \left.\frac{1}{f'(x)}\right|_{x=1}$$

$$= \left.\frac{1}{3x^2+1}\right|_{x=1}$$

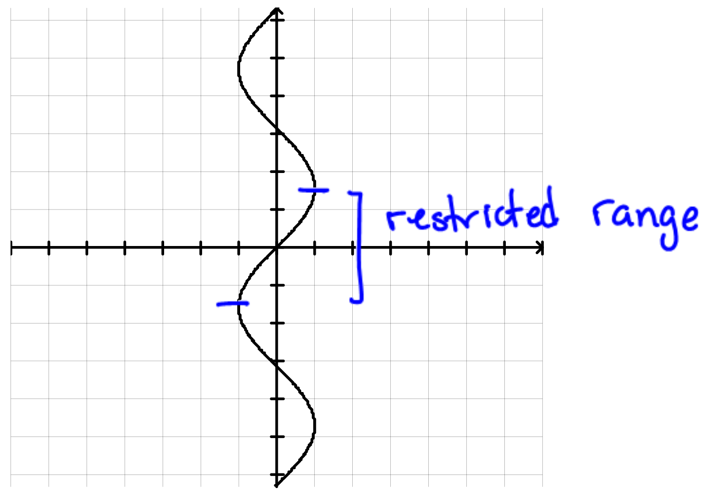
$$= \frac{1}{4}$$

The Inverse Trigonometric Functions

$y = \sin x$

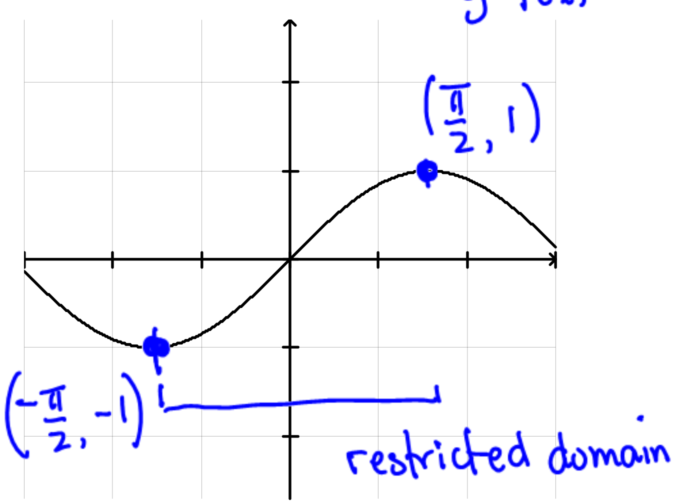


$x = \sin y$

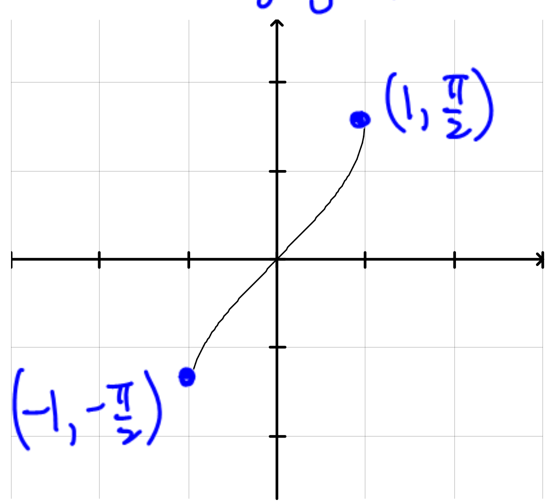


Note that the inverse of $y = \sin x$ is not a function. (ie it is not one-to-one; it does not pass the horizontal line test) To make $y = \sin^{-1} x$ (or $y = \arcsin x$) into a function, we restrict the domain of the original function so that it will pass the horizontal line test. In the case of $y = \sin x$, we can restrict the domain to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

$y = f(x)$



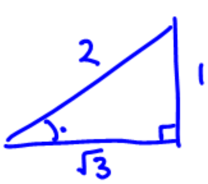
$y = g(x) = f^{-1}(x)$



It is helpful to think of $\sin^{-1}(x)$ as the angle whose sine is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

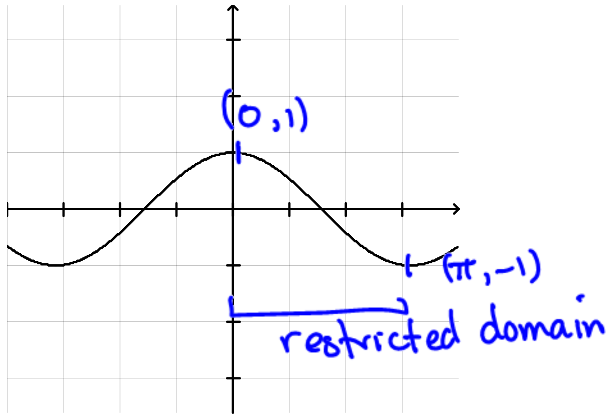
Thus $\sin^{-1}\left(\frac{1}{2}\right) =$ $30°$ or $\frac{\pi}{6}$

$\arcsin\left(-\frac{1}{2}\right) = -30°$ or $-\frac{\pi}{6}$

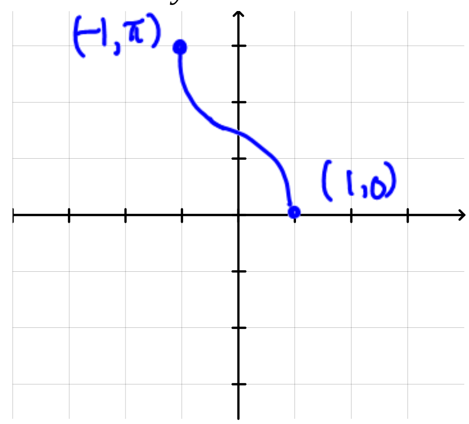


The other inverse trigonometric functions are restricted using the same principle of insuring that the restricted domain produces a one-to-one function.

$y = \cos x$

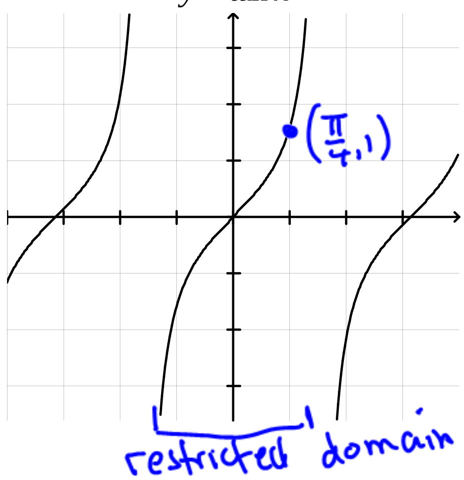


$y = \cos^{-1} x$

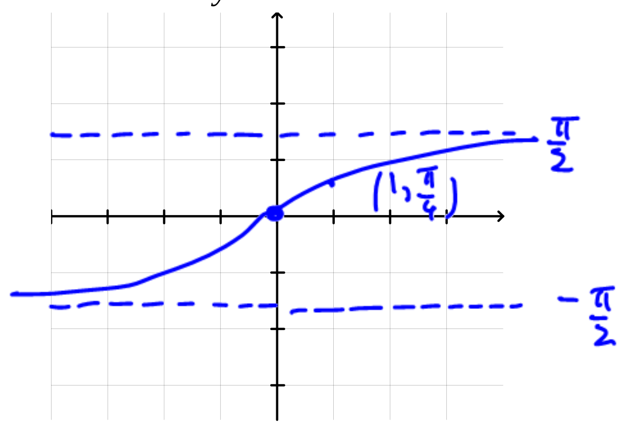


$\cos^{-1}(x)$ is the angle whose $\underline{\cos\theta = x}$ where $\underline{0 \le \theta \le \pi}$

$y = \tan x$

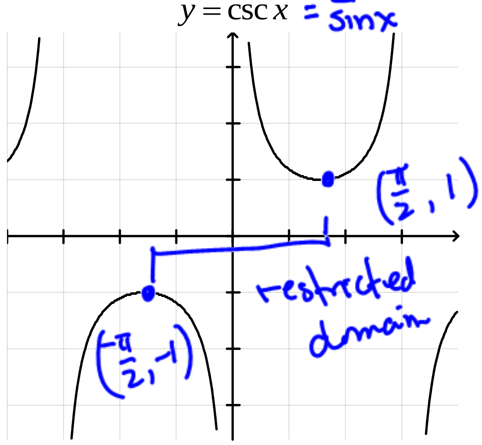


$y = \tan^{-1} x$

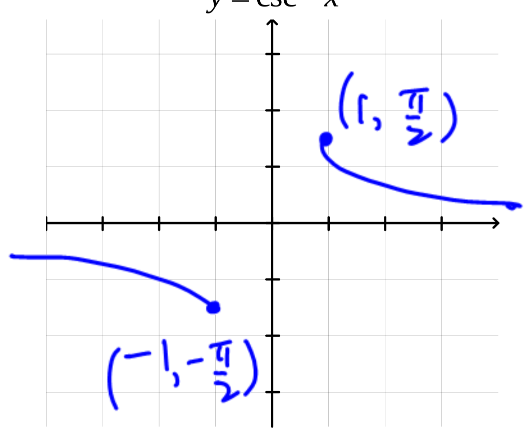


$\tan^{-1}(x)$ is the angle whose $\underline{\tan\theta = x}$ where $\underline{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$

$y = \csc x = \frac{1}{\sin x}$

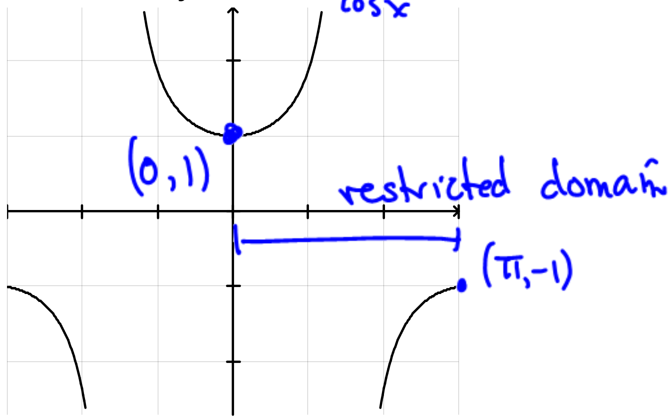


$y = \csc^{-1} x$

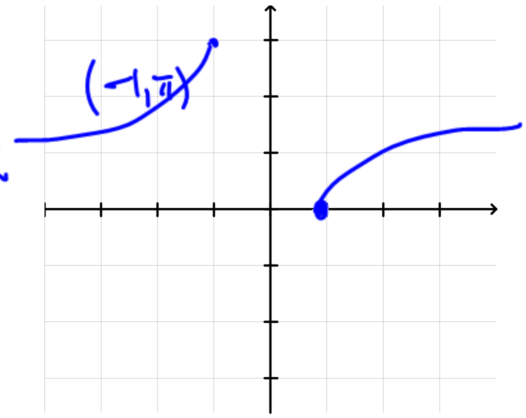


$\csc^{-1}(x)$ is the angle whose $\underline{\csc\theta = x}$ where $\underline{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$

$y = \sec x = \frac{1}{\cos x}$

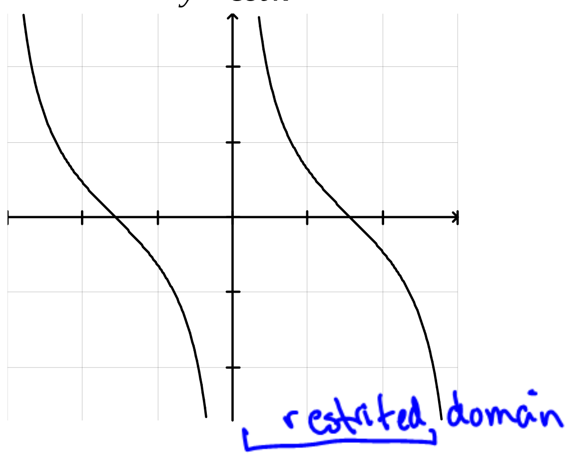


$y = \sec^{-1} x$

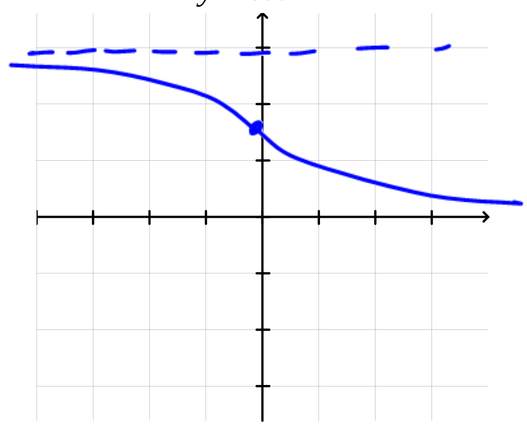


$\sec^{-1}(x)$ is the angle whose $\sec(\theta) = x$ where $0 < \theta < \pi$

$y = \cot x$



$y = \cot^{-1} x$



$\cot^{-1}(x)$ is the angle whose $\cot(\theta) = x$ where $0 < \theta < \pi$

Graphing Calculator Identities

$\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$

$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$

eg $\csc^{-1}(5)$

what θ gives $\csc(\theta) = 5$

or what θ gives $\sin(\theta) = \frac{1}{5}$

find $\dfrac{d\left(\cos^{-1}\left(\frac{1}{2}\right)\right)}{dx}$

$\left(\frac{1}{2}, \_\right)$ is on $\cos^{-1}(x)$

$\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is on $\cos(x)$

$$= \frac{1}{\frac{d(\cos x)}{dx}}\Bigg|_{x=\frac{\pi}{3}} = \frac{1}{-\sin(x)}\Bigg|_{x=\frac{\pi}{3}} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$$

$\boxed{\text{p 162 \# 20, 1–5, 7, 10}}$