11.3A Pascal's Triangle

The triangle below is called Pascal's triangle. Generate the next two rows in the triangle. What patterns do you observe?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

* symmetrical.
* starts/ends with 1
* 2nd # is the row # − 1

Using the version of Pascal's triangle below:

Determine the sum of the numbers in each of the rows. What do you notice?

each row is double previous row, powers of 2

Cover all the multiples of 7, then all the multiples of 5, then all the multiples of 3. What do you notice?

triangles inside the triangle.

1 → 1
1 1 → 2
1 2 1 → 4
1 3 3 1 → 8
1 4 6 4 1 → 16
1 5 10 10 5 1 → 32
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

Cover all the even multiples. What do you notice?

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                                        1
                                     1     1
                                  1     2     1
                               1     3     3     1
                            1     4     6     4     1
                         1     5    10    10     5     1
                      1     6    15    20    15     6     1
                   1     7    21    35    35    21     7     1
                1     8    28    56    70    56    28     8     1
             1     9    36    84   126   126    84    36     9     1
          1    10    45   120   210   252   210   120    45    10     1
       1    11    55   165   330   462   462   330   165    55    11     1
    1    12    66   220   495   792   924   792   495   220    66    12     1
 1    13    78   286   715  1287  1716  1716  1287   715   286    78    13     1
1  14    91   364  1001  2002  3003  3432  3003  2002  1001   364    91    14     1
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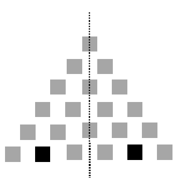
Pascal's Triangle and Combinations

1 1 1 1 2 1 1 3 3 1 1 4 6 4 1	${}_0C_0$ ${}_1C_0$ ${}_1C_1$ ${}_2C_0$ ${}_2C_1$ ${}_2C_2$ ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$ ${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$

The **Symmetrical Pattern** ${}_5C_1$ = ${}_5C_4$

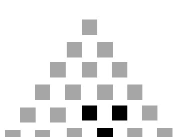


Justification

choosing 1 is the same as excluding 4

Generalization ${}_nC_r = {}_nC_{n-r}$

The **Recursive Pattern** ${}_5C_3$ = ${}_4C_2 + {}_4C_3$



${}_5C_3$

Justification

each item is the sum of 2 items in previous row above it.

Generalization

${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$

1. How would you use Pascal's triangle to determine ${}_{8}C_{5}$?

n r

Recursive Pattern

$_8C_5 = {}_7C_4 + {}_7C_5$

9th row - shows combinations of 8 objects.

6th item - shows when you choose 5 objects.

2. The 6th term in the 14th row in Pascal's triangle is 1287. Express this number as a combination.

$_{13}C_5$

$_{13}C_5 = 1287$

The coefficients in a binomial expansion can be determined using Pascal's triangle.

Binomial	Expansion	Row
$(x+y)^0$	1	1
$(x+y)^1$	$1x+1y$	2
$(x+y)^2$	$1x^2+2xy+1y^2$	3
$(x+y)^3$	$1x^3+3x^2y+3xy^2+1y^3$	4
$(x+y)^4$	$1x^4+4x^3y+6x^2y^2+4xy^3+1y^4$	5

coefficients

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

What do you notice about the powers of x?

start at exponent, decrease by 1 until the last term

What do you notice about the powers of y?

start at 0, increase by 1

What do you notice about the sum of the powers of x and y in each term of the expansion?

add up to make the exponent

What do you notice about the number of terms in the expansion?

exponent + 1

Example 1. Use Pascal's triangle to write the expansion of $(p+q)^6$. ← 7th row

$$1p^6 + 6p^5q + 10p^4q^2 + 15p^3q^3 + 10p^2q^4 + 6p^1q^5 + 1q^6$$

Express the expansion of $(p+q)^6$ using combinations.

$${}_6C_0\,p^6 + {}_6C_1\,p^5q^1 + {}_6C_2\,p^4q^2 + {}_6C_3\,p^3q^3 + {}_6C_4\,p^2q^4 + {}_6C_5\,pq^5 + {}_6C_6\,q^6$$

How does this relate to $(p+q)(p+q)(p+q)(p+q)(p+q)(p+q)$?

it is the same thing, but expanded.

Example 2. Expand $(\underbrace{2x}_{}\underbrace{-5y}_{})^3$

$a = 2x \quad b = -5y \qquad (a+b)^3$

$${}_3C_0\,a^3 + {}_3C_1\,a^2b^1 + {}_3C_2\,ab^2 + {}_3C_3\,b^3$$

$$1(2x)^3 + 3(2x)^2(-5y)^1 + 3(2x)(-5y)^2 + 1(-5y)^3$$

$$8x^3 - 60x^2y + 150xy^2 - 125y^3$$

p542 #1-6, 8-11, 13, 14