Warmup 5.6 #1-6 HW Check webpage for solutions.

Evaluate the following indefinite integrals:

1. $\int x\left(2-5x^2\right)^4\,dx$ $u = 2-5x^2$ $\frac{du}{dx} = -10x$ $dx = \frac{-1}{10x}\,du$ $= \int \cancel{x}\,(u)^4 \cdot \frac{-1}{10\cancel{x}}\,du$ $= -\frac{1}{10}\cdot\frac{1}{5}u^5 + c$ $= -\frac{1}{50}\left(2-5x^2\right)^5 + c$	6. $\int \frac{3e^{2x}}{2+e^{2x}}\,dx$ $u = 2+e^{2x}$ $\frac{du}{dx} = 2e^{2x}$ $dx = \frac{1}{2e^{2x}}\cdot du$ $= \int \frac{3\cancel{e^{2x}}}{u}\cdot\frac{1}{2\cancel{e^{2x}}}\cdot du$ $= \frac{3}{2}\int \frac{1}{u}\cdot du$ $\boxed{= \frac{3}{2}\ln\left(2+e^{2x}\right) + c}$
2. $\int x\sin\left(2x^2\right)dx$ $-\frac{1}{4}\cos\left(2x^2\right) + c$	7.* $\int \sin^3 x\,dx$ $u = \cos x$ $\frac{du}{dx} = -\sin x$ $dx = \frac{-1}{\sin x}\,du$ $\int \sin x\left(\sin^2 x\right)dx$ $\int \sin x\left(1-\cos^2 x\right)dx$ $\int \sin x - \sin x\cos^2 x\,dx$ $-\cos x - \int \cancel{\sin x}\cdot u^2\cdot\frac{-1}{\cancel{\sin x}}\,du$ $-\cos x + \frac{1}{3}u^3 + c$ $\boxed{-\cos x + \frac{1}{3}\cos^3(x) + c}$
3. $\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \cancel{x}\,u^{-\frac{1}{2}}\cdot -\frac{1}{2\cancel{x}}\,du$ $u = 1-x^2$ $\frac{du}{dx} = -2x$ $dx = -\frac{1}{2x}\,du$ $= -\frac{1}{2}\cdot\frac{2}{1}u^{\frac{1}{2}} + c$ $\boxed{= -\left(1-x^2\right)^{\frac{1}{2}} + c}$	8. $\int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)}$ $u = 1+\sqrt{x}$ $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $dx = \frac{2}{1}x^{\frac{1}{2}}\,du$ $= \int \cancel{x^{-\frac{1}{2}}}\,\frac{1}{u}\cdot\frac{2}{1}\cdot\cancel{x^{\frac{1}{2}}}\,du$ $= 2\ln u + c$ $= 2\ln\left(1+\sqrt{x}\right) + c$
4. $\int \frac{1}{\sqrt{1-x^2}}\,dx$ $\sin^{-1}(x) + c$	9. $\int \frac{dx}{9+x^2} = \int \frac{1}{9\left(1+\frac{x^2}{9}\right)}\,dx$ $= \frac{1}{9}\int \frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx$ $u = \frac{x}{3}$ $\frac{du}{dx} = \frac{1}{3}$ $dx = 3\,du$ $= \frac{1}{9}\int \frac{1}{1+u^2}\cdot 3\,du$ $\boxed{= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c}$
5. $\int \frac{dx}{\cos^2(2x)} = \int \frac{1}{\cos^2(u)}\cdot\frac{1}{2}\,du$ $u = 2x$ $\frac{du}{dx} = 2$ $dx = \frac{1}{2}\,du$ $= \int \sec^2 u\cdot\frac{1}{2}\,du$ $\boxed{= \frac{1}{2}\tan(2x) + c}$	10. $\int \frac{3x^2}{\sqrt{x^3}}\,dx = \int \frac{3x^{4/2}}{x^{3/2}}\,dx$ $= \int 3x^{\frac{1}{2}}\,dx$ $= 3\cdot\frac{2}{3}x^{\frac{3}{2}} + c$ $= 2x^{3/2} + c$ $u = x^3$ $\frac{du}{dx} = 3x^2$ $dx = \frac{1}{3x^2}\,du$ $\int 3x^2\cdot u^{-\frac{1}{2}}\cdot\frac{1}{3x^2}\,du$ $\int u^{-\frac{1}{2}}\,du$ $\frac{2}{1}u^{\frac{1}{2}} + c$ $\frac{2}{1}\left(x^3\right)^{1/2} + c$ $2x^{3/2} + c$

Newton's Law of Cooling

Recall that the only solutions to the differential equation $\frac{dy}{dt} = k\,y$ are $y = y_0 e^{kt}$

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings (provided that this difference is not too large) This can be illustrated with the following differential equation:

T = temperature

$$\frac{dT}{dt} = k(T - T_s)$$

$$\int \frac{1}{T - T_s}\,dT = \int k\,dt$$

T_s constant.

$$\ln(T - T_s) = kt + c$$

$$T - T_s = ce^{kt}$$

$T(0) = T_0$

$$T_0 - T_s = C \cdot e^{k(0)}$$

$$C = T_0 - T_s$$

$$\underbrace{T - T_s}_{\text{temp diff at time "t"}} = \underbrace{(T_0 - T_s)}_{\text{Initial temperature diff.}} e^{kt}$$

1. Suppose that an object takes 40 minutes to cool from 30^0 C to 24^0 C in a room that is kept at 20^0 C.
   a) What was the temperature of the object 15 minutes after it was 30^0 C ?
   b) How long will it take the object to cool down to 21^0 C ?

$T_s = 20°$

$T_0 = 30$

$T = 24$

$t = 40$

find "k"

$$T - T_s = (T_0 - T_s)e^{kt}$$

$$24 - 20 = (30 - 20)e^{40k}$$

$$4 = 10e^{40k}$$

$$\frac{4}{10} = e^{40k}$$

$$\ln\frac{4}{10} = 40k \quad \therefore k = -.02291$$

a) $T - 20 = (30 - 20)e^{-.02291(15)}$

$$T = 10e^{(-.02291)(15)} + 20$$

$$T = 27.09\,°C$$

b) $(21 - 20) = (30 - 20)e^{-.02291\,t}$

$$1 = 10e^{-.02291\,t}$$

$$\ln\left(\tfrac{1}{10}\right) = -.02291 \cdot t$$

$t = 100.5$ minutes.

2. A roast turkey is taken from an oven when its temperature has reached 185^0 F and is placed on a table in a room where the temperature is 75^0 F.
   a) If the temperature of the turkey is 150^0 F after half an hour, what is the temperature after 45 minutes?
   b) When will the turkey have cooled to 100^0 F?

$$T - T_s = (T_0 - T_s)e^{kt}$$

$$150 - 75 = (185 - 75)e^{k(30)}$$

$$\frac{75}{110} = e^{30k}$$

$$\ln\frac{75}{110} = 30k$$

$$k = -0.01277$$

a) $T - 75 = (185 - 75)e^{-0.01277 \times 45}$

$$T = 136.92°$$

b) $100 - 75 = (185 - 75)e^{-.01277\,t}$

$$25 = 110\,e^{-0.01277t}$$

$$\ln\frac{25}{110} = -0.01277\,t$$

$t = 116.02$ minutes

3. On a hot day, a thermometer is taken outside from an air conditioned room where the temperature is 21^0 C. After one minute it reads 27^0 C and after 2 minutes it reads 30^0 C
   a) What is the outdoor temperature?
   b) Sketch the graph of the temperature function.

after 1min

$(27-T_s) = (21-T_s)e^{1k}$

$\frac{27-T_s}{21-T_s} = e^k$

after 2min

$(30-T_s) = (21-T_s)e^{2k}$

$\frac{30-T_s}{21-T_s} = e^{2k} = (e^k)^2$

$\frac{30-T_s}{21-T_s} = \left(\frac{27-T_s}{21-T_s}\right)^2$

$(30-T_s)\cancel{(21-T_s)} = \cancel{(21-T_s)}(27-T_s)^2$

$630 - 51T_s + \cancel{T_s^2} = 729 - 54T_s + \cancel{T_s^2}$

$3T_s = 99 \qquad T_s = 33$

the outdoor temperature is 33° C.

4. An investor puts \$100 000 into a bank which pays 5% annual interest compounded continuously. She plans to withdraw money continuously from the account at the rate of \$6000 per year. If $A(t)$ is the amount of money at time t, then

$$\frac{dA}{dt} = .05A - 6000$$

   a) Solve this equation for *A(t)*. Show your work.
   b) When will the money run out?

$\frac{dA}{dt} = .05(A - 120000)$

$\frac{1}{A-120000}dA = .05\,dt$

$\ln(A-120000) = .05t + c$

$A - 120000 = e^{.05t + c}$

$A - 120000 = C \cdot e^{.05t}$

$100\,000 - 120000 = C(1)$

$C = -20\,000$

initial value

$A(0) = 100\,000$

$\boxed{A - 120000 = -20000\, e^{.05t}}$

5. At the age of 20,you make a deposit of \$10 000 to your RRSP which is invested at 6% per year, compounded continuously. You intend to deposit money continuously at a rate of \$3000 per year. Assuming that the rate of interest remains 6%, the amount of money $A(t)$ at time t satisfies the equation:

$$\frac{dA}{dt} = 0.06A + 3000$$

   a) Solve this equation, and determine the amount of money in the your account when you turn 60.
   b) At the age of 60, you decide to withdraw money continuously at the rate of \$15 000 per year. How long will the money in the account last?
   c) If the money is to last you until you are 80, what is the largest possible amount you could withdraw annually?