Integration by Substitution

Some complicated integration problems can often be transformed to simpler ones by using a technique called ***substitution***. This process requires using the following formula (where u is a function of x)

$$\int\left[f(u)\,\frac{du}{dx}\right]dx = \int f(u)\;du$$

This method consists of doing the following

1. * Make a choice for u, say $u = g(x)$

2. Compute $\dfrac{du}{dx} = g'(x)$

3. Make the substitution $u = g(x)$, $du = g'(x)\,dx$

   Note: At this point the entire integral should be in terms of u, no x's should remain.

4. Evaluate the resulting integral.

5. Replace u by $g(x)$

Examples.

1. $\int 3x^2\left(x^3 + 3\right)^{20}\;dx$ $= \int 3x^2\,(u)^{20}\cdot\frac{1}{3x^2}\,du$

$u = x^3+3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{1}{3x^2}\cdot du$

$= \int u^{20}\,du$

$= \frac{1}{21}u^{21} + C$

$= \frac{1}{21}(x^3+3)^{21} + C$

2. $\int \dfrac{dx}{\left(\frac{1}{4}x - 5\right)^4}$ $= \int \dfrac{4du}{u^4}$

$u = \frac{1}{4}x - 5$

$\frac{du}{dx} = \frac{1}{4}$

$dx = 4\,du$

$= \int 4\cdot u^{-4}\,du$

$= 4\cdot -\frac{1}{3}\cdot u^{-3} + C$

$= -\frac{4}{3}\left(\frac{1}{4}x-5\right)^{-3} + C$

3. $\int \sqrt{5-8t}\; dt = \int \sqrt{u} \cdot -\frac{1}{8}\, du$

$u = 5-8t$

$\frac{du}{dt} = -8$

$dt = -\frac{1}{8}\, du$

$= \int u^{\frac{1}{2}} \cdot -\frac{1}{8}\, du$

$= \frac{2}{3} \cdot u^{\frac{3}{2}} \cdot -\frac{1}{8} + C$

$= -\frac{2}{24} u^{3/2} + C$

$= -\frac{1}{12} (5-8t)^{3/2} + C$

4. $\int \frac{2-5y}{\sqrt{5y^2-4y}}\, dy = \int \frac{-5y+2}{\sqrt{u}} \cdot \frac{1}{10y-4}\, du$

$u = 5y^2 - 4y$

$\frac{du}{dy} = 10y - 4$

$dy = du \cdot \frac{1}{10y-4}$

$= \int -\frac{1}{2} \cdot u^{-1/2}\, du$

$= -\frac{1}{2} \cdot \frac{2}{1} u^{\frac{1}{2}} + C$

$= -(5y^2-4y)^{\frac{1}{2}} + C$

5. $\int x^2 \sqrt{x-1}\; dx = \int (u+1)^2\, u^{\frac{1}{2}}\, du$

$u = x-1$

$x = u+1$

$\frac{du}{dx} = 1$

$= \int (u^2 + 2u + 1) \cdot u^{\frac{1}{2}}\, du$

$= \int u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\, du$

$= \frac{2}{7} \cdot u^{\frac{7}{2}} + 2 \cdot \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} \cdot u^{\frac{3}{2}} + C$

$= \frac{2}{7}(x-1)^{\frac{7}{2}} + \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$

6. $\int \frac{(3+y)^2}{5\sqrt{y}}\, dy$

* substitution not necessary

① exponent multiplies nicely

② denominator is a monomial

$= \int \frac{9 + 6y + y^2}{5\sqrt{y}}\, dy$

$= \int \frac{9}{5} y^{-\frac{1}{2}} + \frac{6}{5} y^{\frac{1}{2}} + \frac{1}{5} y^{\frac{3}{2}}\, dy.$

$= \frac{9}{5} \cdot \frac{2}{1} y^{\frac{1}{2}} + \frac{6}{5} \cdot \frac{2}{3} y^{\frac{3}{2}} + \frac{1}{5} \cdot \frac{2}{5} y^{\frac{5}{2}} + C$

$= \frac{18}{5} y^{\frac{1}{2}} + \frac{12}{15} y^{\frac{3}{2}} + \frac{2}{25} y^{\frac{5}{2}} + C$