Exponential Growth and Decay

In many of the sciences, certain quantities grow or decay at a rate that is proportional to their size. Thus if $y = f(t)$ is the value (or amount) of a quantity at time t and if the rate of change of y with respect to t is proportional to its size $f(t)$ at any time t, then $\frac{dy}{dt} = ky$. If we further add the initial condition that $\boxed{y(0) = y_0,}$ what is the solution to this initial value problem?

$$\frac{dy}{dt} = k \cdot y$$
$$\int \frac{1}{y}\, dy = \int k\, dt$$
$$\ln y = kt + c$$
$$y = e^{kt+c}$$

$$y = C \cdot e^{kt}$$
$$y_0 = Ce^{0 \cdot t} \qquad y(0) = y_0$$
$$y_0 = C$$
$$\boxed{\therefore\ y = y_0 e^{kt}}$$

The resulting solution is called the ***Law of Natural Growth*** (k>0) or the ***Law of Natural Decay*** (k<0)
The value k is sometimes called the ***Growth (Decay) Constant***. It is the continuous growth rate (as opposed to growth occurring after discrete time intervals have passed

1. $1000 is invested at 6% compounded continuously. What will this amount grow to in 3 years? What is the effective growth rate?

$$\frac{dA}{dt} = 0.06A$$
$$\frac{1}{A}\, dA = 0.06\, dt$$
$$\ln A = 0.06t + c$$
$$A = e^{0.06t + c}$$

$$y = y_0 e^{kt}$$
$$= 1000 e^{.06(3)}$$
$$\underline{y(3) = 1197.22}$$

effective rate: find 1 year

$$y(1) = 1000 e^{0.06(1)}$$
$$y(1) = 1061.84$$
$$1000(1+r) = 1061.84$$
$$r = 0.06184.$$

2. In the last 2 hours, the number of bacteria in a culture grew from 300 to 800. How many bacteria will there be in the culture 4 hours from now?

$$y = y_0 e^{kt}$$
$$800 = 300 e^{k(2)}$$
$$\frac{8}{3} = e^{2k}$$
$$\ln \frac{8}{3} = 2k$$
$$k = \frac{1}{2} \ln \frac{8}{3}$$
$$k = .490415$$

$$y = y_0 e^{kt}$$
$$y = 800 e^{.490415(4)}$$
$$y = 5688 \text{ bacteria.}$$

3. A population of bacteria grows at a rate of $\boxed{\text{15\% per hour.}}$ If there are presently 10 000 bacteria, how many will there be after 4 hours?

$$y = y_0\left(1 + \frac{r}{n}\right)^{nt}$$

not continuous

$$\therefore\ y = y_0(1+r)^t$$

$$y = 10000\,(1+.15)^4$$

$$= 17490 \text{ bacteria}$$

4. A radioactive substance decays at a rate proportional to the amount present. Determine the half-life of the substance in terms of *k*.

$$\frac{dA}{dt} = kA$$

want the half life.

$$\frac{1}{A}\,dA = k\,dt$$

$$A = Ce^{kt} \quad \text{or} \quad \boxed{A = A_0 e^{kt}}$$

$$\frac{1}{2}\cancel{A_0} = \cancel{A_0}\, e^{kt}$$

$$\frac{1}{2} = e^{kt}$$

$$\ln\frac{1}{2} = kt$$

$$\boxed{t = \frac{1}{k}\cdot\ln\frac{1}{2}}$$

½ life expressed as a function of k.

After 10 years, it is found that a 1.21 mg of a sample of 1000 mg of Carbon-14 has decayed. What is its half-life?

if we find k, we can use the formula

$$y = y_0 e^{kt}$$

$$998.79 = 1000\, e^{k(10)}$$

$$\frac{998.79}{1000} = e^{10k}$$

$$\ln(.99879) = 10k$$

$$k = -1.2107\times10^{-4}$$

or

$$k = -0.00012107$$

$$t = \frac{1}{k}\cdot\ln\left(\frac{1}{2}\right)$$

$$\boxed{t = 5725 \text{ years}}$$

A specimen of charcoal found at Stonehenge contains 63% as much Carbon-14 as a sample of present day charcoal. What is the age of the specimen?

$$y = y_0 e^{kt}$$

$$63 = 100\, e^{(-.00012107)t}$$

$$.63 = e^{(-0.00012107)t}$$

$$\ln .63 = -1.2107\times10^{-4}\, t$$

$$t = 3816 \text{ years old.}$$

5. Suppose that sodium pentobarbital will anesthetize a dog when its bloodstream contains at least 45 mg of sodium pentobarbital per kilogram of body weight of the dog. Suppose also that sodium pentobarbital is eliminated exponentially from a dog's bloodstream, with a half life of 5h. What single dose should be administered to anesthetize a 50-kg dog for 1 h?

$$t = \frac{1}{k}\ln\frac{1}{2}$$

$$5 = \frac{1}{k}\cdot\ln\frac{1}{2}$$

$$k = \frac{\ln\frac{1}{2}}{5}$$

$$k = -.1386$$

$$y = y_0\, e^{k(1)}$$

$$50(45) = y_0\cdot e^{k}$$

$$2250/e^{-.1386} = y_0 \quad = 2584.5\text{mg}$$